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On the irregular refraction of a plane shock wave at a Mach number interface

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The paper considers the refraction of a plane shock wave at an interface between two streams of different Mach number. Particular attention is paid to the irregular wave systems. It is found that when the interface is slow–fast, that is when the speed of sound a_0 in the first or incident wave medium is less than the speed of sound a_B in the second or transmitted wave medium, then there are two irregular systems, one being a double Mach reflexion type and the other being a four-wave confluence type. There are also two irregular systems when the refraction is fast–slow; these are a single Mach reflexion type and an expansion wave type. This last system has a central expansion wave when the flow is steady and a continuous band expansion wave when the flow is self-similar. Only two of the irregular wave systems have been observed experimentally in the fully developed state. Possible degeneracies are discussed.

1. Introduction

A shock wave propagating through a gas will be refracted if it encounters a change in the acoustic impedance§ in the gas. This may be due to a change in some property of the gas such as its temperature or composition. After refraction there will appear a transmitted wave which is always a shock, and a reflected wave which may be either a shock or an expansion wave. Experiments conducted by Jahn (1956) using a gas interface indicated that the observed wave patterns could be broadly classified into regular and irregular systems. A regular system includes those refractions in which all the waves lie along straight rays that emanate from a well-defined refraction point. Any system that does not have this property is considered to be irregular. An alternative classification which is often useful is based on differences in the speeds of sound in the gas or gases. Thus if the incident shock i is in a part of the gas where the speed of sound is a_0 and the transmitted shock t is in another part where it is a_B then the refraction is called 'slow–fast' if $a_0 < a_B$ and 'fast–slow' if $a_0 > a_B$. The nomenclature is illustrated in figure 1.

The theory of regular refraction as devised by Polachek & Seeger (1951), Henderson (1966, 1967), and others, is multi-valued. The solutions may be con-

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§ Defined here as ρa. If ρa of the medium changes then the shock will refract except perhaps for very restricted initial conditions. A more general definition has been given by Polachek & Seeger (1951).

veniently arranged as an ordered set in which the ordering is determined by the strength or pressure ratio across the transmitted shock. If the set is compared with Jahn's data then it is found that it is always the weakest member that has appeared. This is also the minimum entropy solution. The theory of the irregular

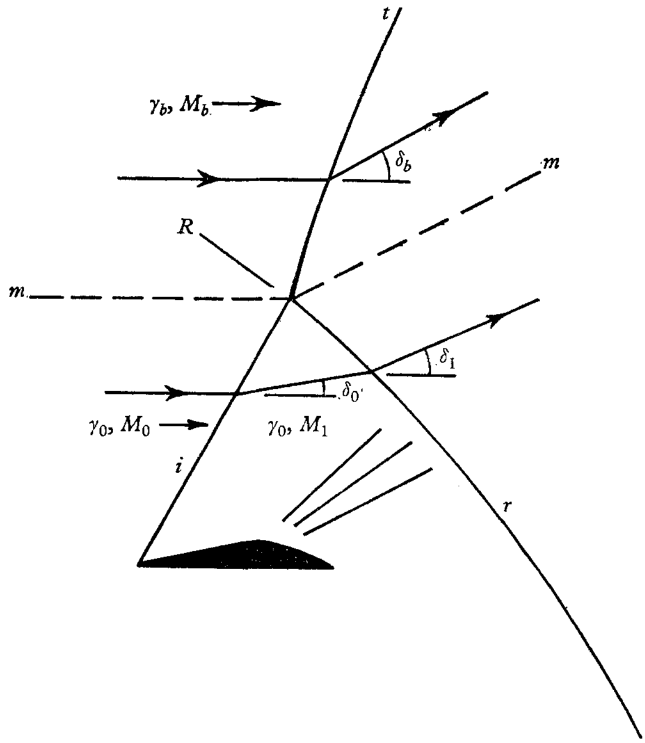

FIGURE 1. Refraction at a Mach number and gas discontinuity: *mm*, interface; *R*, refraction point; *i*, incident shock; *r*, reflected shock; *t*, transmitted shock.

wave systems is not so complete, but with the help of a simple mapping technique some qualitative deductions can be made about the structure of these systems. It is also possible to give precise quantitative conditions for the appearance of any particular type of system. The objective of the present paper is to discuss the refraction of a plane shock wave at the interface between two streams of different Mach number. Special attention will be paid to the irregular phenomena in an attempt to refine and extend previous results. In an earlier paper (Henderson 1966) initial conditions were found for an irregular system of unknown form. The nature of this wave system is discussed here and it is shown to be a single Mach reflexion variety. The paper concludes with an atlas of wave refraction systems.

2. Regular refraction

2.1. *Analysis*

Using methods that were described in the earlier papers, numerical results were obtained from a computer for the regular refraction of a plane shock wave at a Mach number interface. This consisted of two parallel streams of air having the

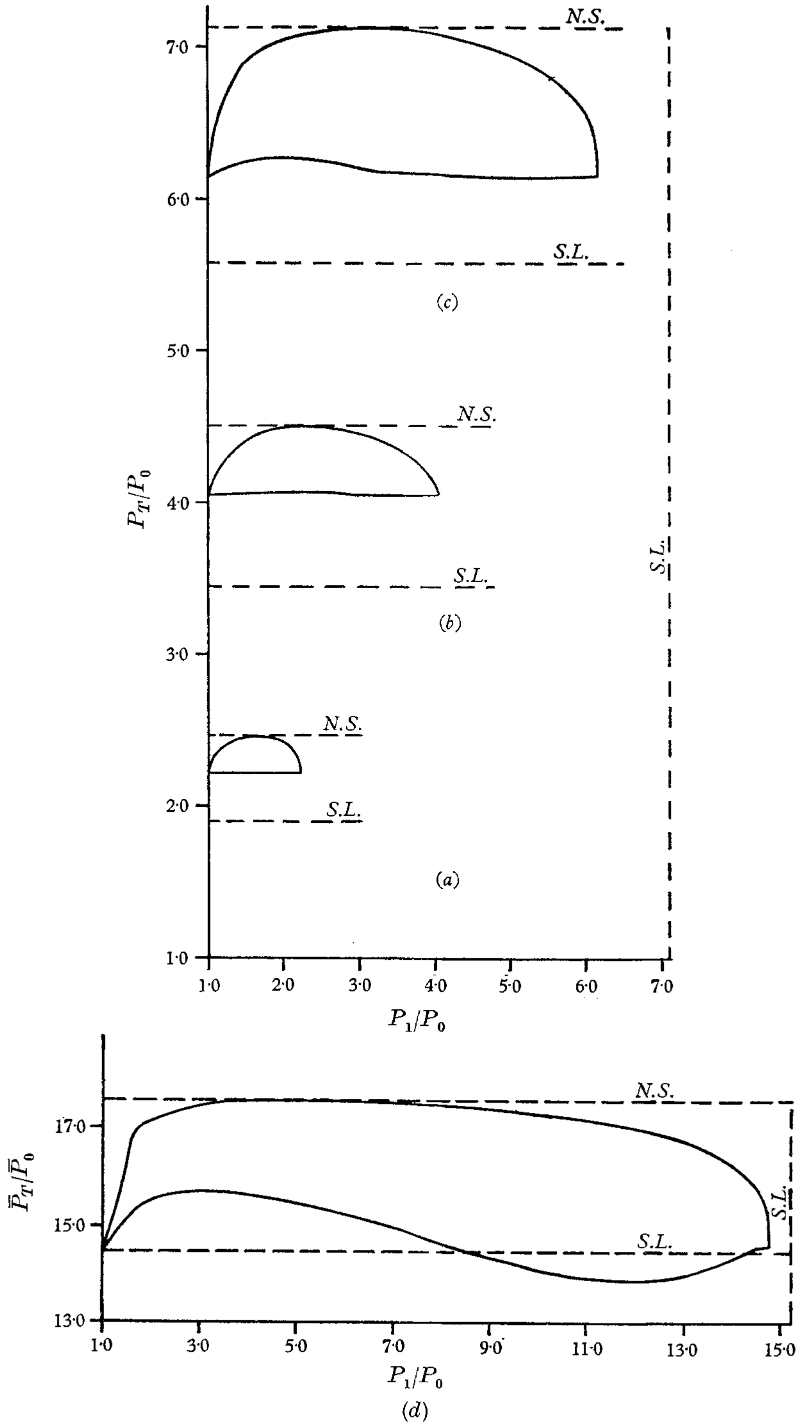

FIGURE 2. Physically significant roots of the polynomial equation for the regular refraction of a plane shock at a Mach number discontinuity when $M_B < M_0$, —polynomial root line; *S.L.*, sonic line; *N.S.*, normal shock line. (*a*) $M_0 = 3{\cdot}0$, $M_B = 1{\cdot}5$; (*b*) $M_0 = 3{\cdot}0$, $M_B = 2{\cdot}0$; (*c*) $M_0 = 3{\cdot}0$, $M_B = 2{\cdot}5$; (*d*) $M_0 = 4{\cdot}0$, $M_b = 3{\cdot}9$.

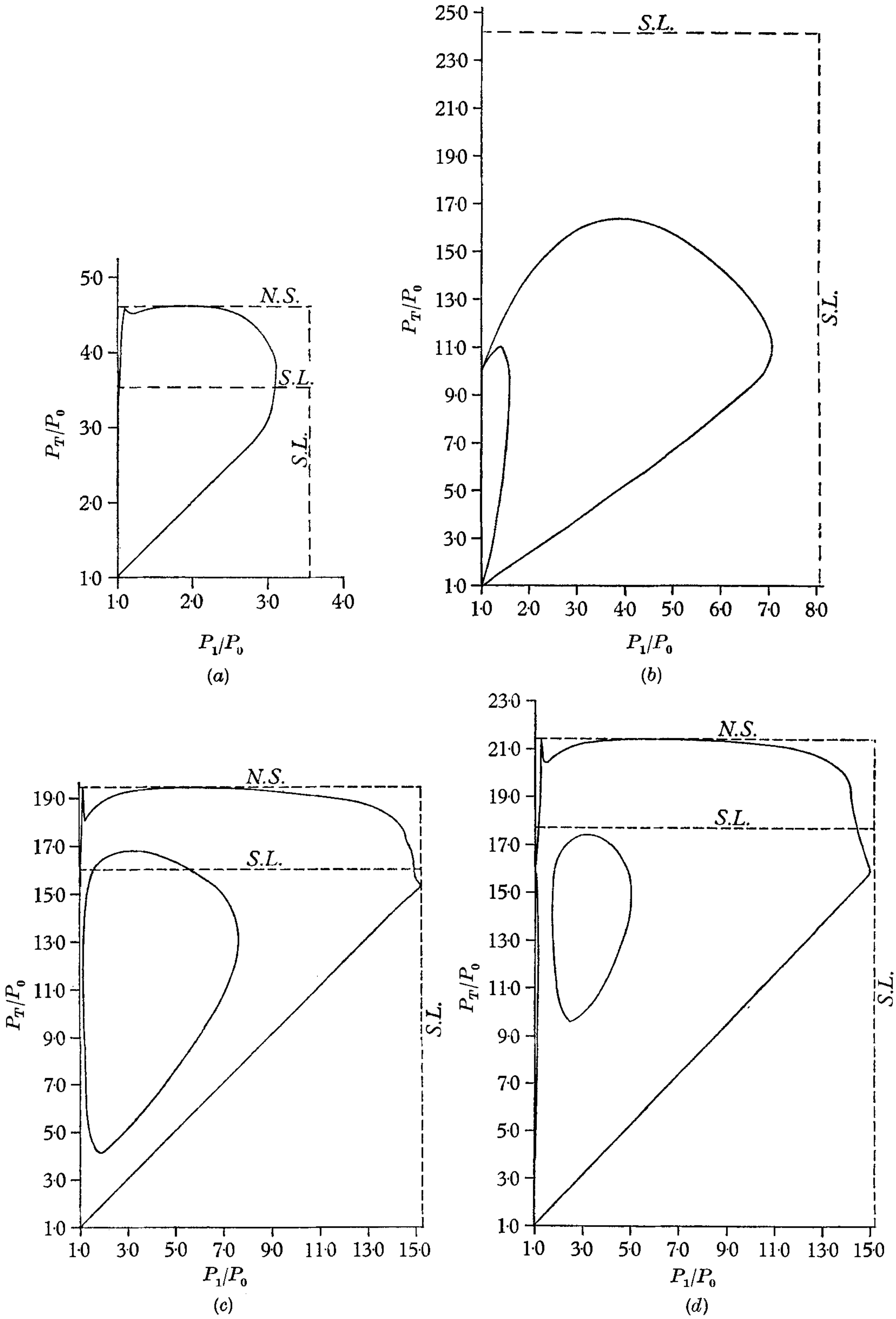

FIGURE 3. Physically significant roots of the polynomial equation for the regular refraction of a plane shock at a Mach number discontinuity when $M_B > M_0$; —, polynomial root line; *S.L.*, sonic line; *S.N.*, normal shock line. (*a*) $M_0 = 2{\cdot}0$, $M_B = 2{\cdot}025$; (*b*) $M_0 = 3{\cdot}0$, $M_B = 5{\cdot}0$; (*c*) $M_0 = 4{\cdot}0$, $M_B = 4{\cdot}1$; (*d*) $M_0 = 4{\cdot}0$, $M_B = 4{\cdot}3$.

same stagnation enthalpy. Such an interface may be found when an initially homogeneous and adiabatic stream is subsequently required to follow different entropy paths in different parts of the field. Practical examples are found in the flow associated with multi-shock intakes, and in the refraction of shock waves in wing wakes. The following relation is then valid

$$\frac{a_0^2}{a_B^2} = \frac{T_0}{T_B} = \frac{1+\frac{1}{2}(\gamma-1)M_B^2}{1+\frac{1}{2}(\gamma-1)M_0^2}. \tag{1}$$

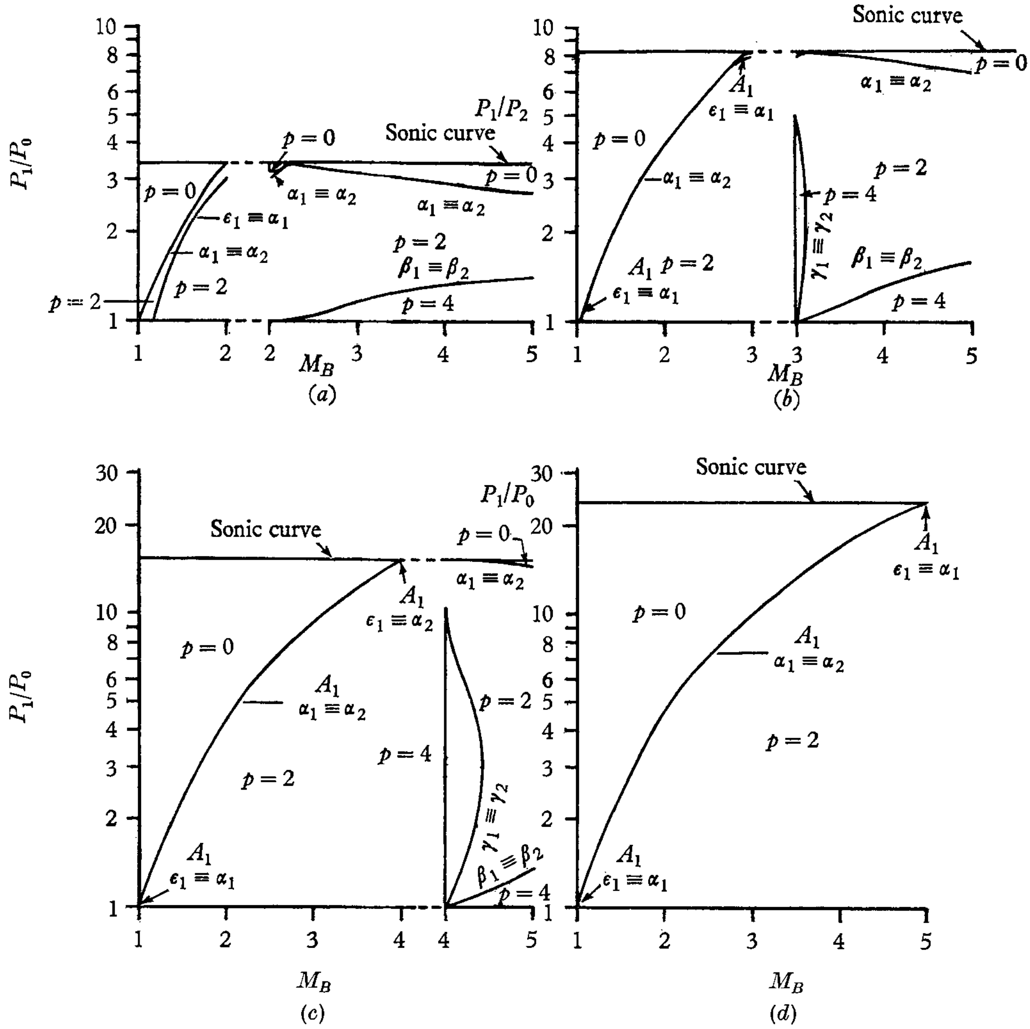

FIGURE 4. Multiple root lines and other critical curves. Division indicates change from slow–fast $M_0 > M_B$ to fast–slow $M_B < M_0$. (*a*) $M_0 = 2$; (*b*) $M_0 = 3$; (*c*) $M_0 = 4$; (*d*) $M_0 = 5$. p, number of physically significant roots.

The results for several Mach numbers are shown in figures 2–4. The graphs are of two types, one being a plot of the pressure ratio of the transmitted versus incident shocks and the other being a series of critical curves delineating boundaries between various regions of initial conditions.

The equations of regular refraction can also be solved graphically by constructing the hodograph diagram. This is a powerful method for investigating the physical consequences of the numerical results. The procedure will be to suppose that the polar and/or characteristic for the reflected wave are initially coincident with the polar for the incident shock. The reflected wave maps are then gradually and continuously displaced until all solutions of physical interest cease to exist. This is physically equivalent to commencing with the incident shock as a Mach line and then slowly increasing its strength, causing at first a regular refraction to appear and then continuing the development until an irregular refraction appears in its place. In this way a variety of phenomena can be arranged in an orderly sequence.

2.2. *Slow–fast interface*

To begin with it will be assumed that the incident shock i is in the M_0 stream and the refraction is initially slow–fast and further that $M_B < M_0 < \sqrt{2}$. In these circumstances polar II, defined by M_B = constant, is entirely contained or nested inside of polar I, M_0 = constant, figure 5. If the polar for the reflected shock III, M_1 = constant, is slightly displaced from I then the ordered set obtained is (α_1, α_2). With continued displacement the two solutions eventually come into coincidence $(\alpha_1 \equiv \alpha_2)$ and with still further displacement they become unreal. The set of regular solutions is now empty (0) and from previous experience it is to be expected that the physical result will be that an irregular wave system appears in place of a regular one. The double solution is thus a boundary that separates regions of initial conditions corresponding to regular and irregular wave systems. Some numerical results are shown in figure 4.

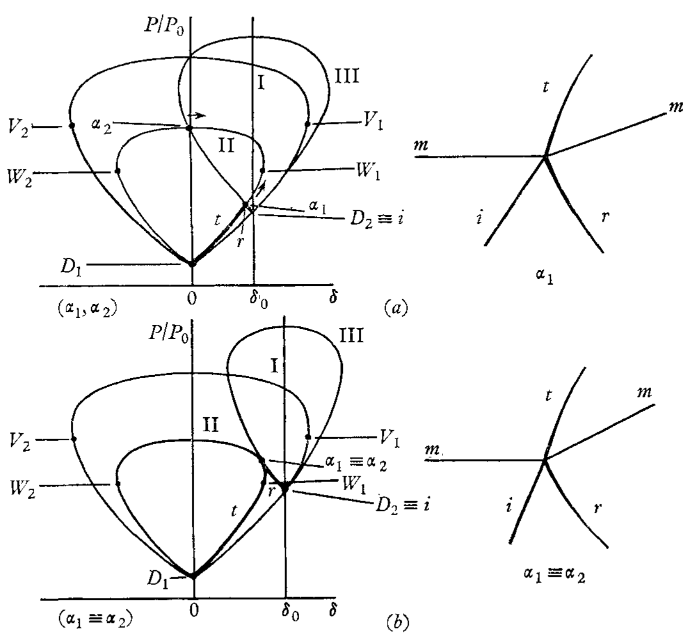

FIGURE 5 (*a*) and (*b*). For legend see p. 191.

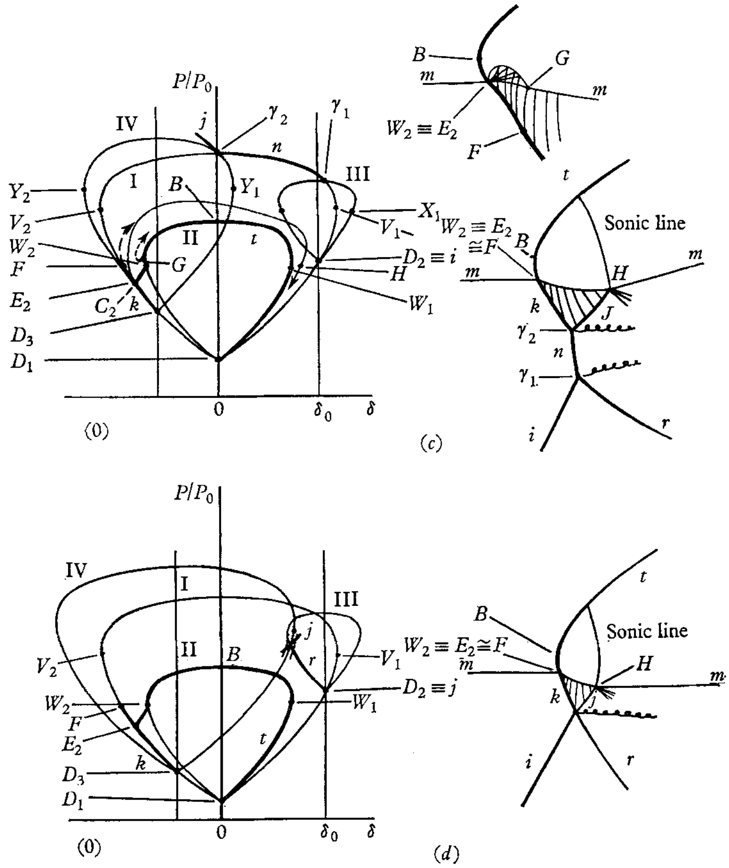

FIGURE 5. Slow–fast sequence when polar III is displaced from polar I and polar II contained inside polar I.

At higher Mach numbers polar II is found to intersect polar I. The intersection points $A_{1,2}$ may be found by the method given in the appendix and it is shown that for given M_0 the minimum value of M_B that will give intersections that have physical significance is given by

$$M_{B\,\min} \equiv M_0(M_0^2-1)^{-\frac{1}{2}}. \tag{2}$$

In particular there can be no real intersections unless $M_0 \geqslant \sqrt{2}$. Once $A_{1,2}$ do exist then polar II will lie partly outside of polar I, figure 6. Displacement now causes the ordered set to be $(\epsilon_1, \alpha_1, \alpha_2)$. The ϵ_1 solution is obtained by displacing the characteristic c_1 and represents a regular refraction with a reflected expansion wave e. With continued development ϵ_1 and α_1 approach A_1 until there is the coincidence $D_2 \equiv \epsilon_1 \equiv \alpha_1 \equiv A_1$. The physical result is that the reflected wave degenerates to a Mach line for both ϵ_1 and α_1. The degenerate set may be denoted $(\epsilon_1 \equiv \alpha_1, \alpha_2)$. With further development ϵ_1 ceases to exist but the reflected wave of the α_1 solution strengthens and again becomes a shock (α_1, α_2). The set finally becomes empty (0) beyond the coincidence $\alpha_1 \equiv \alpha_2$. It may be assumed on the basis of previous experience and also on arguments presented in earlier papers that it is always the weakest member of the set that appears. This is subject to the provision that all downstream boundaries must be removed to infinity. It follows that the degeneracy $(\epsilon_1 \overset{A_1}{\equiv} \alpha_1, \alpha_2)$ is a limiting condition which separates a region of initial conditions where a regular refraction has a reflected expansion from one where the reflected wave is a shock. The condition is easily computed

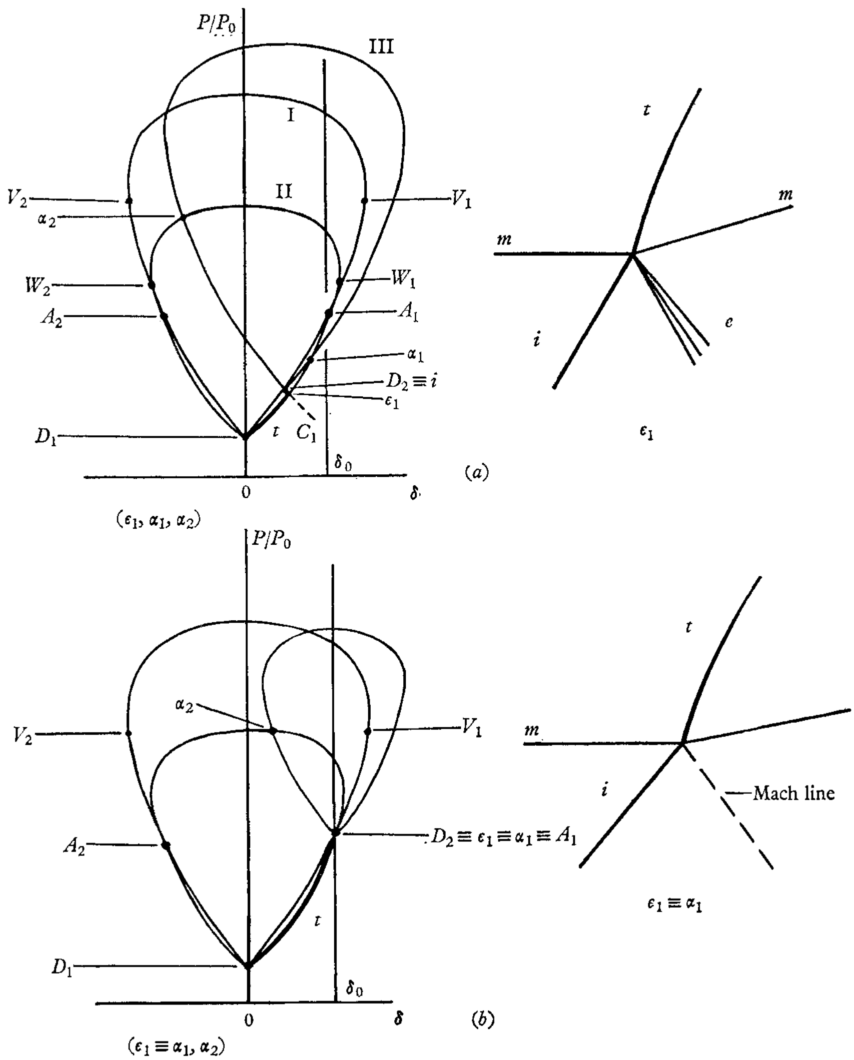

FIGURE 6 (*a*) and (*b*). For legend see p. 194.

from the equation given in the appendix and some results are shown in figure 4. Transitions in the ordered set such as

$$(\epsilon_1, \alpha_1, \alpha_2), \quad (\epsilon_1 \overset{A_1}{\equiv} \alpha_1, \alpha_2), \quad (\alpha_1, \alpha_2),$$

which occur at a polar intersection point, have a different character from one such as:

$$(\alpha_1, \alpha_2); \quad (\alpha_1 \equiv \alpha_2); \quad (0),$$

which occurs at no well-distinguished point. Physically the former is associated with a Mach line degeneracy of the reflected wave and the nature of this wave is different on either side of the transition. With the latter there is no degeneracy and the reflected wave remains of finite strength and, moreover, the wave system is regular on one side of the transition and irregular on the other. If now the Mach numbers are further increased but the restriction $M_0 > M_B$ is still retained then differences appear in the irregular wave systems, figure 7; discussion of them is deferred to a later section.

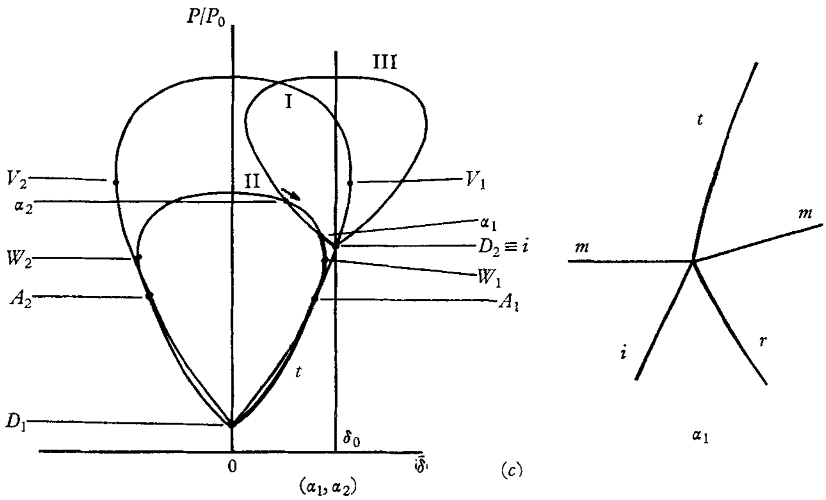

FIGURE 6 (c). For legend see p. 194.

Ordered set	Transition	Remark
$(\alpha_1, \beta_1, \beta_2, \alpha_2)$		
	$\beta_1 \equiv \beta_2$	—
(α_1, α_2)		
	$\gamma_1 \equiv \gamma_2$	—
$(\alpha_1, \gamma_1, \gamma_2, \alpha_2)$		
	$\gamma_1 \equiv \gamma_2$	—
(α_1, α_2)		
	$\epsilon_1 \overset{A_1}{\equiv} \alpha_1$	Polar intersection point
$(\epsilon_1, \alpha_1, \alpha_2)$		
	$\alpha_1 \equiv \alpha_2$	—
(ϵ_1)		
	$D_2 \equiv V_1$	Sonic point
(0)		Irregular wave system

TABLE 1. Example of a sequence of ordered sets and their transitions

2.3. *Fast–slow interface*

The refraction is fast–slow when $M_0 < M_B$. The first sequence, figure 8, is for the condition $M_0 < M_B < \sqrt{2}$. Polar I is nested inside polar II and this is the reciprocal case of figure 5. The set has only one member (ϵ_1) and it is valid over the range of initial conditions defined by the end points $D_2 \equiv D_1$ and $D_2 \equiv V_1$. These correspond respectively to a Mach line degeneracy of i and a sonic degeneracy of e, i.e. $M_1 = 1$. At higher Mach numbers the situation is more complicated and a typical sequence of sets with their transitions is given symbolically in table 1.

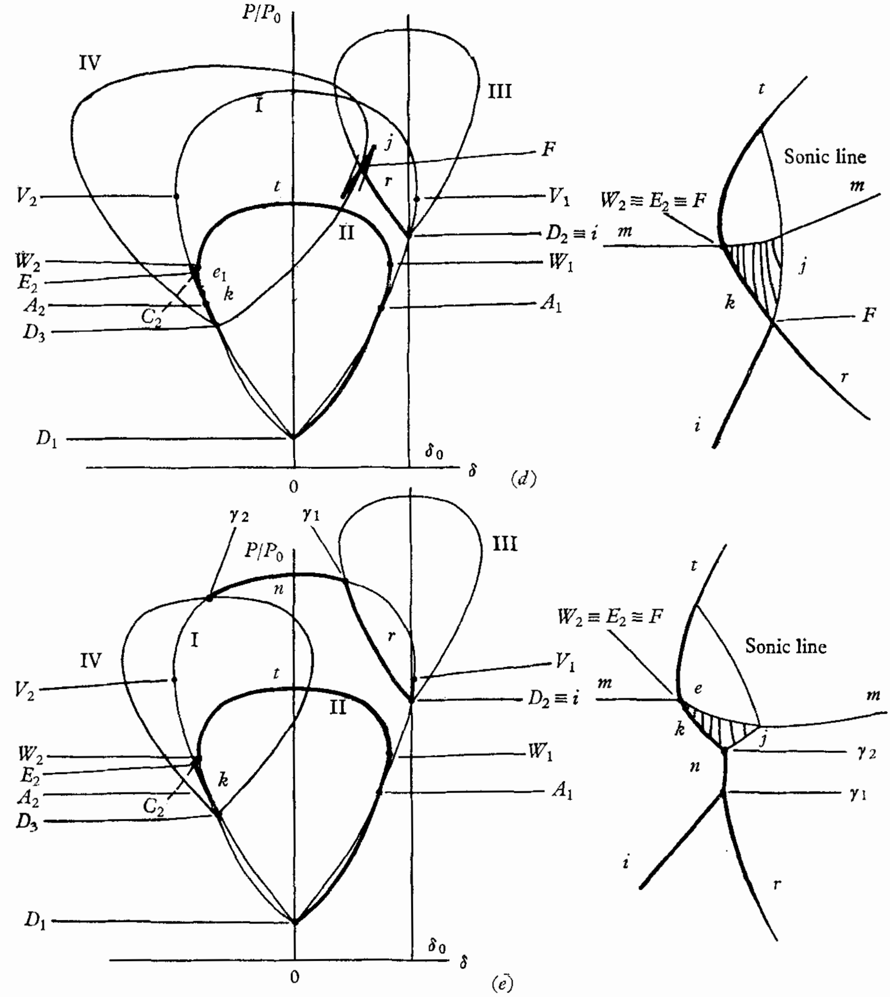

FIGURE 6. Slow–fast sequence when polar III is displaced from polar I, where intersections A_1, A_2 exist between polars I, II and where ord. $A_{1,2}$ < ord. $W_{1,2}$.

Similar sequences have been discussed in earlier papers so the diagrams are omitted, but some numerical results are shown in figure 4. One effect of further increases in the Mach numbers $M_{0,B}$ is to restrict the range of conditions for the appearance of the $\gamma_{1,2}$ solutions and in fact they may become unreal if M_B becomes large enough. Another effect is that the ϵ_1 solution ceases to exist and so for these reasons the sequence tends to become simpler as the Mach numbers increase still further. Differences also appear in the irregular wave system figure 9), and these will be discussed in the next section.

3. Irregular refraction

3.1. *Slow–fast interface*

Consider again the slow–fast sequence shown in figure 5 and it is now desired to find out what happens when the ordered set becomes empty. In the hodograph plane a gap is opened in the interior of polar I (Guderley 1947, 1962; Kawamura & Saito 1956). Although the maps are very similar to ones that have been discussed in the earlier papers, the present discussion is aimed at exploring the situation more deeply. Above the interface, t maps into the segment D_1W_2, where W_2 is one of the sonic points of polar II. For this sequence, figure 5*c*, there are no physically significant intersections between polars I and II so following Guderley they are joined by a characteristic† which begins at W_2. The point E_2 requires the existence of a shock occurring at the Mach number M_0. This wave, which is labelled k, will in general be curved and it maps into the segment E_2FD_3 on polar I; D_3 represents the strength of k at the confluence γ_2. Polar IV is erected at D_3 and for simplicity it is constructed for the average Mach number $\langle M_2 \rangle$ downstream of k. If there is no intersection between polars III and IV then a Mach stem n will be present, figure 5*c*; in this case the entire wave system will be called a double Mach reflexion type. If the polars do intersect then the Mach stem will be absent and a four-wave confluence type of system will appear in its place as indicated in figure 5*d*. The latter system has been photographed by Jahn at a gas interface, and because hodograph maps for both the gas and the Mach number interfaces are qualitatively identical it is concluded that the same type of wave system will appear at both interfaces. The transition illustrated in figure 5*c*, is analogous to the Mach reflexion system at the outlet of an over-expanded nozzle changing into a four-wave system as the nozzle pressure ratio is increased.

The hodograph diagram shows that the streamlines converge in the region W_2E_2F. This indicates that in the physical plane the primary refraction point $W_2 \equiv E_2$ will be a point of infinite streamline curvature (see Guderley 1947; Sternberg 1959). The region is in the negative δ half-plane which means that the local streamlines are deflected downwards. Further along the interface the streamlines approach the positive δ half-plane and an inflexion G is induced in the physical plane. The expansion fan W_2E_2 is oriented in the first family of characteristics and a Guderley patch is indicated for the physical plane. Its sonic line apparently terminates on the interface at, or near, G. In the photographs taken by Jahn there was no sign of the region $GBW_2 \equiv E_2F$ although in some photographs the shock segment BW_2 was visible. This latter event only happened when a wall boundary was placed close to the interface. The shock t then consisted entirely of BW_2 and terminated on the wall at B (Henderson 1966). It is concluded that G is too close to the primary refraction point for the region to be detected except as a shock segment.

† In an earlier paper it was concluded that the appropriate characteristic to use was c_1 which is in the opposite family to c_2. However, after a more detailed study it was decided that c_2 was more physically satisfactory. There is little chance of deciding the matter by experiment because on the basis of Sternberg's (1959) work it would be expected that the expansion fans corresponding to $c_{1,2}$ would be too small to detect. This means that there would be no visible effect on the wave system if the solutions were interchanged.

If the free-stream Mach numbers $M_{0,B}$ are now increased then eventually the polar intersections $A_{1,2}$ will appear. Initially they will be on the supersonic parts of the polars but this development does not affect the irregular wave system in

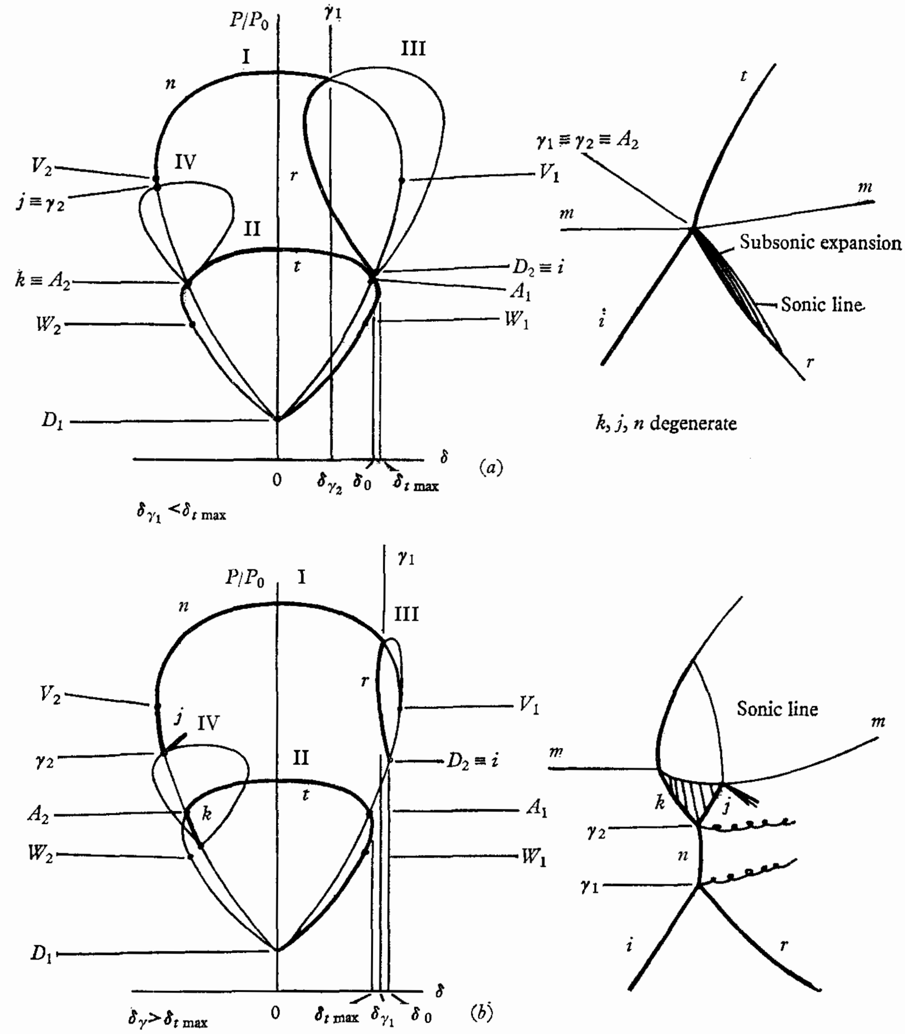

FIGURE 7. Slow–fast sequence when polar III is displaced from polar I where ord. $A_{1,2}$ > ord. $W_{1,2}$.

any fundamental way. As $M_{0,B}$ increase still further $A_{1,2}$ approach the sonic points $W_{1,2}$ and then coincide with them $A_{1,2} \equiv W_{1,2}$. During this process E_2 is required to move towards the sonic point V_2 and this will cause polar IV to shrink steadily. If initially a four wave confluence is present then it may be forced to split into a double Mach reflexion system. With a still further increase in $M_{0,B}$ it is found that ord. $A_{1,2}$ > ord. $W_{1,2}$ and $A_{1,2}$ will lie on the subsonic part of polar II (figure 7). When Jahn photographed the wave system corresponding to this condition there was no sign of the shocks k, j or n. Since the hodograph diagram indicates that they should be present it can only be surmised that they are confined to the immediate vicinity of the refraction point. This may be called a geometric degeneracy in the sense where a line is shrunk into a point. It is possible however to force the wave n to appear. In a previous paper (Henderson 1966) it was found that the condition necessary to make n grow was that $\delta_\gamma > \delta_{l\,\max}$. Such a condition can often be realized merely by increasing the strength of i, thus compare figures 7a and 7b. There was still no sign of k and j in Jahn's photographs but in principle it should be possible to find conditions that force them to grow. These waves will only disappear from the maps when $A_{1,2}$ lie on the subsonic part of polar I but for the circumstances considered here this does not happen.

3.2. *Fast–slow interface*

Suppose now the refraction is fast–slow. The first sequence, figure 8, is for the condition where polar I is nested inside polar II; this is the reciprocal condition of figure 5, and one has $M_0 < M_B < \sqrt{2}$. The irregular wave system appears as

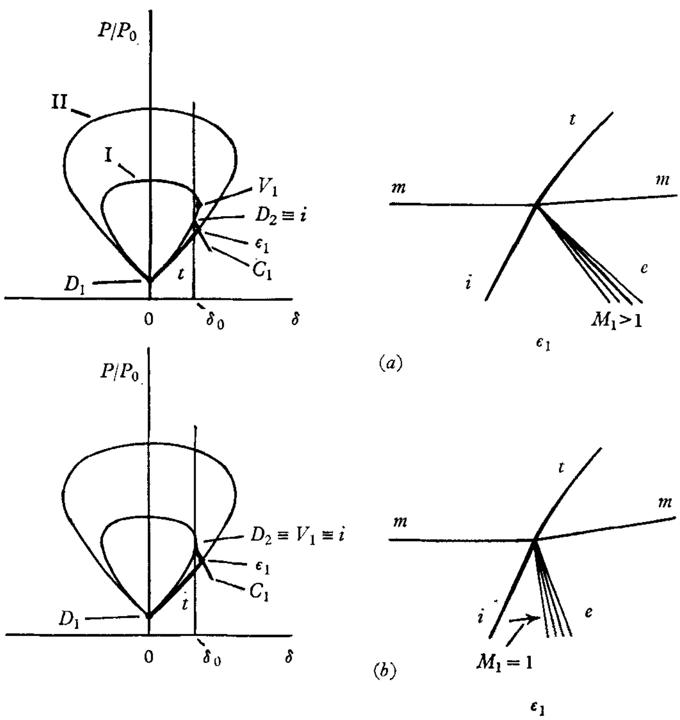

FIGURE 8 (a) and (b). For legend see page 198.

soon as the set (ϵ_1) becomes empty (0), that is as soon as D_2 is displaced beyond the sonic point V_1. Jahn's photographs of the corresponding condition at a gas interface showed that the ϵ_1 solution became irregular by the expansion fan broadening into a continuous band of expansion waves as shown in figure 8c. In this case however the hodograph diagram shows that it is necessary to distinguish between self-similar and steady-state solutions. Now Jahn obtained his data from a shock tube where the flow was substantially self-similar and it is this solution that is mapped in figure 8*c*. But when the flow is steady it seems impossible to construct the same type of solution unless there is an undisturbed

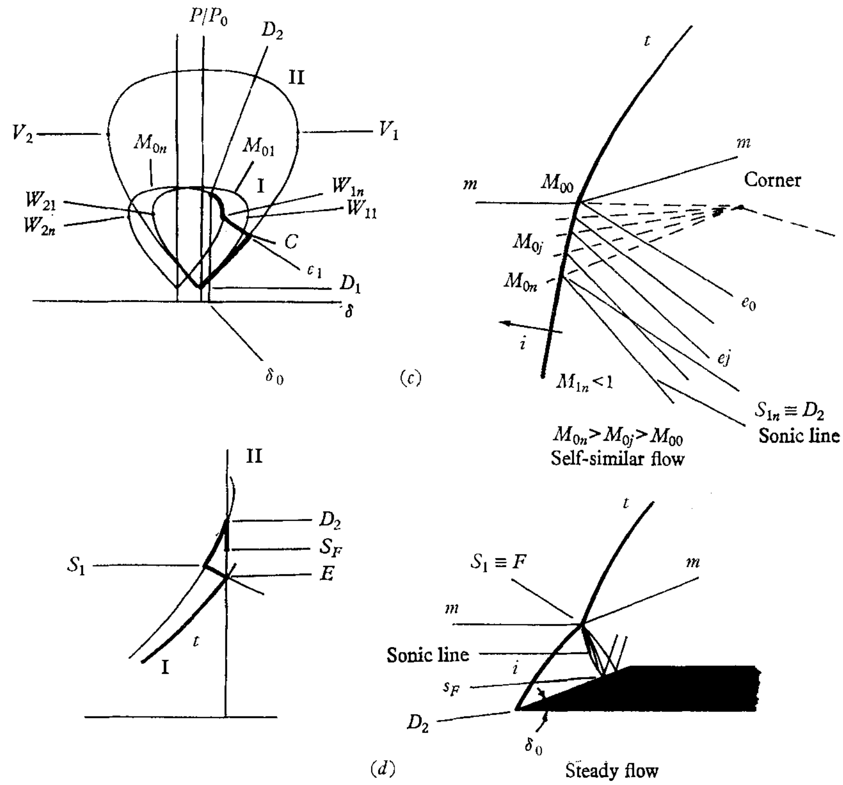

FIGURE 8. Fast–slow sequence when polar II is displaced from polar I. Polar I contained within polar II.

region of gas with suitably varying properties such as to give a similar Mach number distribution. If, however, the undisturbed region has uniform properties then the hodograph diagram shows that the expansion fan should be centred, as in figure 8*d*. The flow downstream of *i* is now subsonic $M_1 < 1$ and will depend on the nature of the disturbance causing *i*; a finite wedge has been chosen for illustration (Henderson 1967). Guderley (1962) has discussed analogous flows in supersonic jets.

At higher Mach numbers the intersections $A_{1,2}$ again appear and initially these will be on the supersonic parts of the polars, ord. $A_{1,2} <$ ord. $V_{1,2}$, but this does not result in any basic change in the irregular systems. With a continuous Mach number increase, $A_{1,2}$ eventually cross the sonic points so that ord. $A_{1,2} >$ ord. $W_{1,2}$ and they then lie on the subsonic part of polar I. There is now a basic change in the character of the hodograph diagram as may be seen in figure 9. In this case the irregular system appears after the formation of the double root $\alpha_1 \equiv \alpha_2$. The hodograph diagram shows that an extra shock is now present and

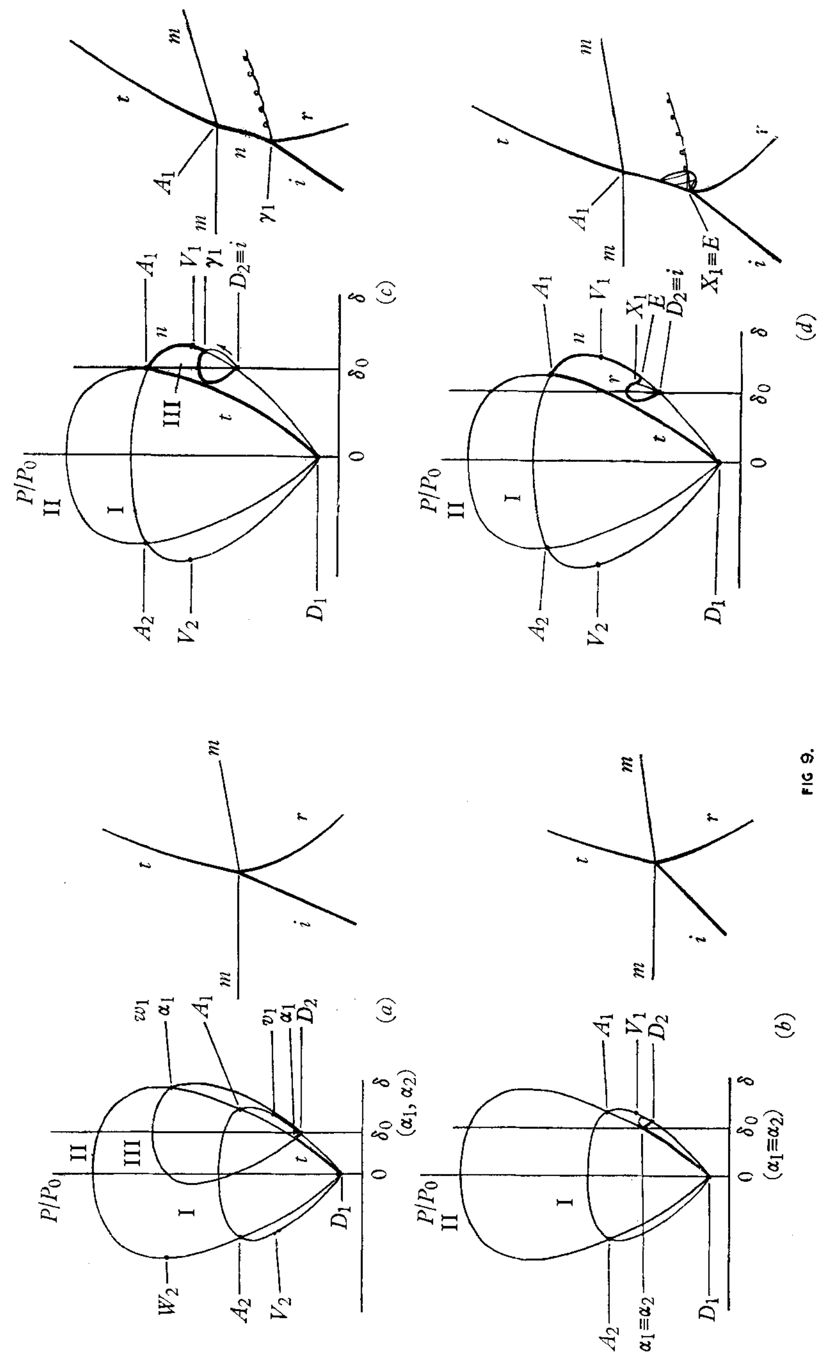

FIGURE 9. Fast–slow sequence when polar III is displaced from polar I, where ord. $A_{1,2}$ > ord. $V_{1,2}$.

this may be interpreted as a Mach stem n. The intersection of the polars I and III means that a three-shock confluence γ is also present. The irregular wave system is thus seen to have the nature of a single Mach reflexion (figure 9c). The flow in this example is supersonic downstream of t and subsonic downstream of n. As development continues $D_2 \to V_1$ and in the process γ moves on to the supersonic parts of polars I and III. This causes a Guderley patch to be present at the confluence (figure 9d). The initial conditions needed for the appearance of a single Mach reflexion system were noted in a previous paper (Henderson 1966) but no attempt was made to predict its structure. Jahn did not perform an experiment at the required initial conditions and to the best of our knowledge it has not yet been observed.

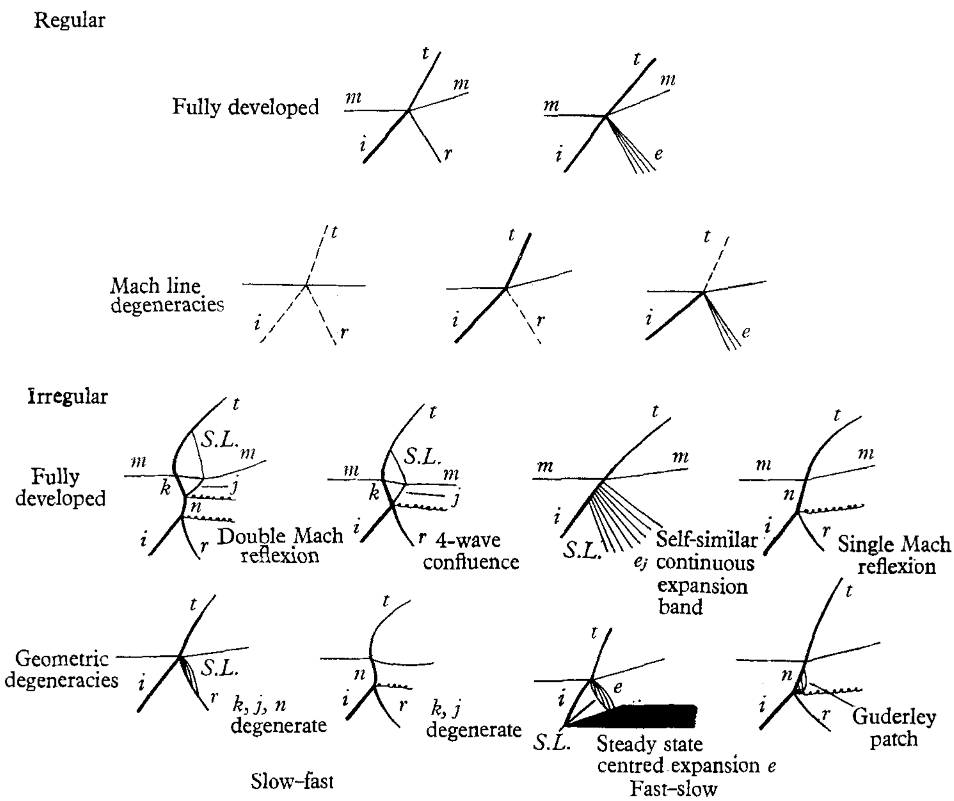

FIGURE 10. Atlas of wave refraction systems.

4. Concluding remarks

It seems well established that only two wave systems are possible when a shock wave undergoes *regular* refraction. These are distinguished by the nature of the reflected wave, being a shock in one case and an expansion in the other. Both types have been observed experimentally in both the slow–fast and fast–slow gas combinations. By contrast all the irregular wave systems are not found in both combinations. Thus for slow–fast gases the irregular wave systems are a double Mach reflexion type and a four-wave confluence type, but for fast–slow gases the irregular systems are a single Mach reflexion type and an expansion wave type. In this last system the expansion wave is centred when the flow is steady and a continuous band when the flow is self-similar. An atlas of wave systems is shown in figure 10. Experimentally the four-wave confluence system has been observed fully developed but the double Mach reflexion type has only been observed in one or other of two degenerate forms, namely with shocks n, k j absent or with shocks k, j absent. The single Mach reflexion system has not yet been observed in refraction, although it is of course well known in reflexion. The continuous band type has been observed fully developed but the centred wave type does not yet seem to have been observed.

The hodograph diagrams for the Mach number interface do not appear to differ in any fundamental way from those of the gas interface and in this respect they are qualitatively identical. It is to be expected that if two wave systems are identical in this sense in the hodograph plane then they will be identical in the same sense in the physical plane. It must be recognized that this does not exclude distortions and geometrical degeneracies so that two qualitatively identical wave systems can have a very different appearance. In this respect we are talking of the topological properties of the wave system which are so deep seated that they persist under almost arbitrary continuous deformations of the system.

Appendix: Determination of the free-stream polar interactions

At the intersection points $A_{1,2}$ of polars I and II one has:

$$\delta_0 = \delta_B; \quad \frac{P_1}{P_0} = \frac{P_T}{P_0} = x. \tag{A 1}$$

Put
$$b = 1+\gamma M^2; \quad d = \frac{2\gamma}{\gamma+1}M^2 - \frac{\gamma-1}{\gamma+1};$$

then from the shock polar equation (Henderson 1966) one has:

$$\frac{d_0 - x}{(b_0-x)^2} = \frac{d_B - x}{(b_B - x)^2}.$$

Therefore

$$x^2[2(b_B-b_0)+d_0-d_B] - x[b_B^2 - b_0^2 + 2(b_B d_0 - b_0 d_B)] + b_B^2 d_0 - b_0^2 d_B = 0;$$

or, substituting and simplifying,

$$x^2 - \frac{\gamma+1}{2}(M_0^2+M_B^2)x - 1 + \gamma M_0^2 M_B^2 - \frac{\gamma-1}{2}(M_0^2+M_B^2) = 0$$

and so
$$x = \frac{\gamma+1}{4}(M_0^2+M_B^2) \pm \frac{1}{2}\left[\left(\frac{\gamma+1}{2}\right)^2 (M_0^2+M_B^2)^2 - 4\gamma M_0^2 M_B^2 + 2(\gamma-1)(M_0^2+M_B^2)+4\right]^{\frac{1}{2}}. \tag{A 2}$$

Equation (A 2) gives the ordinates of the polar intersection points $A_{1,2}$. In the special case when $A_{1,2}$ coincide with the double point D_1 then $x = 1$ and equation (A 2) gives the result

$$M_B = \frac{M_0}{(M_0^2-1)^{\frac{1}{2}}}.$$

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